NORMALIZATION BY EVALUATION

AGENDA:

1. WHAT IS NORMALIZATION?
2. DERIVING A NORMALIZER FROM AN INTERPRETER
3. ADDING TYPED EQUALITY

$$
\begin{aligned}
e ::=\ & x \quad \mid \mathrm{rec}_A(e,e,e) \\
\mid\ & \lambda x.\, e \\
\mid\ & e\, e \\
\mid\ & \underline{n}
\end{aligned}
$$

$$
\begin{aligned}
v ::=\ & \boxed{x.e \mid P} \\
\mid\ & \uparrow^A ne \\
\mid\ & \underline{n}
\end{aligned}
$$

$$
\begin{aligned}
ne ::=\ & x \quad \mid \mathrm{rec}_A(ne, no, no) \\
\mid\ & ne\ no
\end{aligned}
$$

$$no ::= \downarrow^A v$$

$$
\begin{aligned}
A, B ::=\ & \mathbb{N} \\
\mid\ & A \to B
\end{aligned}
$$

$$\frac{\Gamma, x : A \vdash e_1 : B \qquad \Gamma \vdash e_2 : A}{\Gamma \vdash (\lambda x.\, e_1)\, e_2 = [e_2/x]e_1 : B}\ \beta$$

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash f = \lambda x.\, f\, x : A \to B}\ \eta$$

$$\frac{\Gamma \vdash b : A \qquad \Gamma \vdash s : \mathbb{N} \to A \to A}{\Gamma \vdash \mathrm{rec}_A(\underline{0}, b, s) = b : A}\ r_1$$

$$\frac{\cdots}{\Gamma \vdash \mathrm{rec}_A(\underline{n+1}, b, s) = (s\ \underline{n})(\mathrm{rec}_A(\underline{n}, b, s)) : A}\ r_2$$

$$[\![-]\!]_\rho$$

$$R(\Gamma\ -)$$

$[\![x]\!]_\rho = \rho(x)$ $[\![-]\!]_- : e \to \rho \to v$

$[\![\lambda x.\, e]\!]_\rho = \boxed{x.e \mid \rho}$

$[\![e_1\ e_2]\!]_\rho = ap([\![e_1]\!]_\rho, [\![e_2]\!]_\rho)$

$[\![\underline{n}]\!]_\rho = \underline{n}$

$[\![rec_A(t, b, s)]\!]_\rho = r_A([\![t]\!]_\rho, [\![b]\!]_\rho, [\![s]\!]_\rho)$

$ap(\boxed{x.e \mid \rho}, v) = [\![e]\!]_{\rho, x \mapsto v}$ $ap : v \to v \to v$

$ap(\uparrow^{A \to B}_{ne}, v) = \uparrow^B(ne\ (\downarrow^A v))$

$r_A(\underline{0}, b, s) = b$ $r_A : v \to v \to v \to v$

$r_A(\underline{n+1}, b, s) = ap(ap(s, n), r_A(n, b, s))$

$r_A(\uparrow^N ne, b, s) = \uparrow^A rec(ne, \downarrow^A b, \downarrow^{N \to A \to A} s)$

$R(\downarrow^{A \to B} v) = \lambda \hat{x}.\, R(ap(v, \hat{x}))$

$R(\downarrow^N \uparrow^N ne) = R_N(ne)$

$R(\downarrow^N \underline{n}) = \underline{n}$

$R : no \to e$

$R_N : ne \to e$

$R_N(x) = x$

$R_N(ne\ v) = R_N(ne)\ R(v)$

$R_N(rec_A(ne, b, s)) = rec_A(R_N(ne), R(b), R(s))$

$Norm(\Gamma, A, e) = R(\downarrow^A [\![e]\!]_{env(\Gamma)})$

$Norm : \Gamma \to A \to e \to e$

$env(\cdot) = \cdot$

$env(\Gamma, x{:}A) = env(\Gamma), x \mapsto \uparrow^A x$

$env : \Gamma \to \rho$