Homotopy theoretic aspects of ~~Constructive Type Theory~~ Reversible Computing

HoTT Π

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PL Wonks
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Problem

What is a *sound* and *complete* model for Π ?

What does *completeness* have to do with *Univalence* ?

A likely solution

Π	**HoTT**	∞ **grpd**	**space**
$\Pi_{\mathbb{2}}$	$\sum_{X:\mathcal{U}} \lVert X = \mathbb{2} \rVert$	Δ_2	$K(S_2, 1) \simeq \mathbb{R}P^\infty$
Π_n	$\sum_{n:\mathbb{N}} \sum_{X:\mathcal{U}} \lVert X = \mathtt{Fin}\ n \rVert$	$\coprod_{n \in \mathbb{N}} \Delta_n$	$\oplus_{n\in\mathbb{N}} K(S_n, 1)$
$\Pi_{1/2}$	$\sum_{X:\mathcal{U}} \lVert X = \mathbb{S}^1 \rVert$	$\mathbf{B}\Delta_2$	??

Slogan I

Internal logic is better than external logic!

Prelude

$$\mathtt{\Omega}\ \dot{T} :\equiv * =_T *$$

$$\mathtt{Aut}\ T :\equiv T \simeq T$$

$$\mathtt{BAut}\ T :\equiv \sum_{X:\mathscr{U}} \| X \simeq T \|$$

$$\{\!\!\{ T \}\!\!\} :\equiv \sum_{X:\mathscr{U}} \| X = T \|$$

$$T_0 :\equiv (T, |\ \mathtt{refl}\ T\ |) : \{\!\!\{ T \}\!\!\}$$

Univalent Fibrations

Let $P : A \to \mathscr{U}$ be a type family (fibration).

Using `transport` along P,

$$f : x =_A y \to P(x) \to P(y)$$

$$g : x =_A y \to P(y) \to P(x)$$

$$\omega : x =_A y \to P(x) \simeq P(y)$$

P is *univalent* if ω is an equivalence.
(Lumsdaine, Kapulkin, Voevodsky)

Slogan II

The identity fibration is univalent!

— *The Univalence Axiom (Voevodsky)*

Univalent Universes

Let $\tilde{\mathtt{U}} \coloneqq (\mathtt{U} : \mathscr{U}, \mathtt{El} : \mathtt{U} \to \mathscr{U})$ be a universe à la Tarski.

$\tilde{\mathtt{U}}$ is *univalent* if $\mathtt{El}$ is a univalent fibration.

Univalent sub-universes

Lemma:

For any $T : \mathscr{U}$, the sub-universe $(\{\!\!\{T\}\!\!\}, \mathsf{fst})$ is univalent.

Corollary:

$$\Omega\ \{\!\!\{T\}\!\!\} \simeq \Omega\ \mathsf{BAut}\ T \simeq \mathsf{Aut}\ T$$

Semantics of $\Pi_{\mathbb{2}}$

Using $T = \mathbb{2}$:

$$\Omega\ \{\![\mathbb{2}]\!\} \simeq \Omega\ \mathtt{BAut}\ \mathbb{2} \simeq \mathtt{Aut}\ \mathbb{2}$$

Characterize $\mathtt{Aut}\ \mathbb{2}$:

$$\mathtt{Aut}\ \mathbb{2} \simeq \mathbb{2}$$

Slogan III

Equivalences are injections!

$\mathsf{Aut}\ \mathbb{2} \simeq \mathbb{2}$

If $f : \mathbb{2} \to \mathbb{2}$ is an equivalence, then f is either `id` or `not`.

$$
\begin{aligned}
& f(\mathsf{true}) = f(\mathsf{false}) \\
\to \quad & \mathsf{true} = \mathsf{false} \\
\to \quad & \bot
\end{aligned}
$$

Soundness & Completeness

$$\begin{tikzcd} \mathsf{Syn} \arrow[rr, bend left, "\mathsf{Con}"] \arrow[rr, phantom, "\perp"] & & \mathsf{Mod} \arrow[ll, bend left, "\mathsf{Lan}"] \end{tikzcd}$$

Slogan IV

Functions are Functors!

Soundness & Completeness

Level 0:

$$[\![\_]\!]_0 : \Pi_{\mathbb{2}} \to \{\![\mathbb{2}]\!\}$$

$$\ulcorner\_\urcorner_0 : \{\![\mathbb{2}]\!\} \to \Pi_{\mathbb{2}}$$

Level 1:

$$[\![\_]\!]_1 : \prod_{\mathtt{A},\mathtt{B}:\Pi_{\mathbb{2}}} (\mathtt{A} \leftrightarrow_1 \mathtt{B}) \to ([\![A]\!]_0 = [\![B]\!]_0)$$

$$\ulcorner\_\urcorner_1 : (\mathbb{2}_0 = \mathbb{2}_0) \to (\ulcorner\mathbb{2}_0\urcorner_0 \leftrightarrow_1 \ulcorner\mathbb{2}_0\urcorner_0)$$

and so on ...

Epilogue

- Checkout our paper:
  - *From Reversible Programs to Univalent Universes and Back*
    `https://arxiv.org/abs/1708.02710`

- Follow our work on GitHub:
  - `vikraman/2DTypes`
  - `rrose1/basic-hott`
  - `ssomayyajula/HoTT`
  - `DreamLinuxer/Pi2`