

*Homotopy theoretic
aspects of
~~Constructive Type Theory~~
Reversible Computing*

HoTT II

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PL Wonks
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Problem

What is a *sound* and *complete* model for Π ?

What does *completeness* have to do with *Univalence* ?

A likely solution

Π	HoTT	∞ grpd	space
Π_2	$\sum_{X:\mathcal{U}} \ X = 2 \ $	Δ_2	$K(S_2, 1) \simeq \mathbb{R}P^\infty$
Π_n	$\sum_{n:\mathbb{N}} \sum_{X:\mathcal{U}} \ X = \mathbf{Fin} \ n \ $	$\coprod_{n \in \mathbb{N}} \Delta_n$	$\bigoplus_{n \in \mathbb{N}} K(S_n, 1)$
$\Pi_{1/2}$	$\sum_{X:\mathcal{U}} \ X = \mathbb{S}^1 \ $	$\mathbf{B}\Delta_2$??

Slogan I

Internal logic is better than external logic!

Prelude

$$\Omega \dot{T} := * =_T *$$

$$\text{Aut } T := T \simeq T$$

$$\text{BAut } T := \sum_{X:\mathcal{U}} \| X \simeq T \|$$

$$\{T\} := \sum_{X:\mathcal{U}} \| X = T \|$$

$$T_0 := (T, | \text{refl } T |) : \{T\}$$

Univalent Fibrations

Let $P : A \rightarrow \mathcal{U}$ be a type family (fibration).

Using transport along P ,

$$f : x =_A y \rightarrow P(x) \rightarrow P(y)$$

$$g : x =_A y \rightarrow P(y) \rightarrow P(x)$$

$$\omega : x =_A y \rightarrow P(x) \simeq P(y)$$

P is *univalent* if ω is an equivalence.

(Lumsdaine, Kapulkin, Voevodsky)

Slogan II

The identity fibration is univalent!

— *The Univalence Axiom (Voevodsky)*

Univalent Universes

Let $\tilde{U} := (U : \mathcal{U}, E1 : U \rightarrow \mathcal{U})$ be a universe à la Tarski.

\tilde{U} is *univalent* if $E1$ is a univalent fibration.

Univalent sub-universes

Lemma:

For any $T : \mathcal{U}$, the sub-universe $(\{T\}, \text{fst})$ is univalent.

Corollary:

$$\Omega \{T\} \simeq \Omega \text{BAut } T \simeq \text{Aut } T$$

Semantics of Π_2

Using $T = \mathbb{Z}$:

$$\Omega \{\{\mathbb{Z}\}\} \simeq \Omega \text{BAut } \mathbb{Z} \simeq \text{Aut } \mathbb{Z}$$

Characterize $\text{Aut } \mathbb{Z}$:

$$\text{Aut } \mathbb{Z} \simeq \mathbb{Z}$$

Slogan III

Equivalences are injections!

Aut 2 \simeq 2

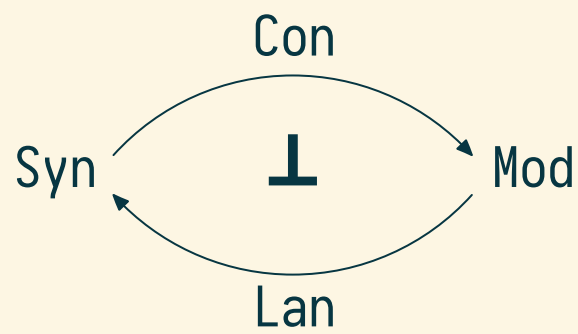
If $f : 2 \rightarrow 2$ is an equivalence, then f is either **id** or **not**.

$$f(\text{true}) = f(\text{false})$$

$$\rightarrow \text{true} = \text{false}$$

$$\rightarrow \perp$$

Soundness & Completeness



Slogan IV

Functions are Functors!

Soundness & Completeness

Level 0:

$$\llbracket - \rrbracket_0 : \Pi_2 \rightarrow \{\mathbb{Z}\}$$

$$\ulcorner - \urcorner_0 : \{\mathbb{Z}\} \rightarrow \Pi_2$$

Level 1:

$$\llbracket - \rrbracket_1 : \prod_{A, B: \Pi_2} (A \leftrightarrow_1 B) \rightarrow (\llbracket A \rrbracket_0 = \llbracket B \rrbracket_0)$$

$$\ulcorner - \urcorner_1 : (\mathbb{Z}_0 = \mathbb{Z}_0) \rightarrow (\ulcorner \mathbb{Z}_0 \urcorner_0 \leftrightarrow_1 \ulcorner \mathbb{Z}_0 \urcorner_0)$$

and so on ...

Epilogue

- Checkout our paper:
 - ▶ *From Reversible Programs to Univalent Universes and Back*
<https://arxiv.org/abs/1708.02710>
- Follow our work on GitHub:
 - ▶ vikraman/2DTypes
 - ▶ rrose1/basic-hott
 - ▶ ssomayyajula/HoTT
 - ▶ DreamLinuxer/Pi2