Universe Hierarchies

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(Generalized) Universe Polymorphism

PL Wonks

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Act I

The Dawn of T~~ime~~ypes

Comprehension, 1901

In naive set theory we have

- $(- \in X)$ for any set X
- for any predicate P the set comprehension $\{x \mid P(x) \text{ holds}\}$

eg from $\mathrm{Even}(n) := 2 \mid n$ we have $\mathrm{Evens} := \{n \mid 2 \mid n\}$; $\mathrm{Even}(n) \iff n \in \mathrm{Evens}$

eg from $\mathrm{NotContainsSelf}(X) := X \notin X$ we have
$\mathrm{SetsNotContainingSelf} := \{X \mid X \notin X\}$.
If $\mathrm{SNCS} \in \mathrm{SNCS}$ then $\mathrm{NotContainsSelf}(\mathrm{SNCS})$ so $\mathrm{SNCS} \notin \mathrm{SNCS}$.
Conversely, $\mathrm{SNCS} \notin \mathrm{SNCS} \Rightarrow \mathrm{SNCS} \in \mathrm{SNCS}$. (Russell 1901)

Restricted Comprehension

Axiom For any set S and predicate P over S we can form $\{s \in S \mid P(s) \text{ holds}\}$.

Thus the sets are stratified by "type" (Russell + Whitehead 1910–, Church 1940):
individuals (ι), propositions (o), predicates over some type ($\iota \to o$, $(\iota \to o) \to o$)

Note The restriction to comprehension is negated if we admit a set of all sets, but this is convenient (eg, category theory) ...

Universes

We can have a collection of all sets; it just can't be a set.

In set theory these are called large cardinal axioms:

- There is a "large set" of all sets. (These are like sets except that comprehension produces large sets, not sets.)
- There is a "huge set" of all large sets.

  ⋮

In type theory we say $\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \cdots$ (and importantly, not $\mathcal{U}_i : \mathcal{U}_i$).
Each $\mathcal{U}_i$ is like $\mathcal{U}_{<i}$ except that $\mathcal{U}_{i-1}$ is also an element.
In particular, $A : \mathcal{U}_{<i} \Rightarrow A : \mathcal{U}_i$.

Act II

Universe Polymorphism

It's Really Unfortunate That $\mathcal{U} : \mathcal{U}$ Is Inconsistent

$$\mathrm{id} : (X : \mathcal{U}_0) \to X \to X$$

"$\forall X. X \to X$"

We can't instantiate X with $\mathcal{U}_0$, so maybe we want

$$\mathrm{id} : (X : \mathcal{U}_1) \to X \to X$$

But now we still can't instantiate X with $\mathcal{U}_1$ …

$$\mathrm{id} : (X : \mathcal{U}_{640000}) \to X \to X$$

"640K universes should be enough for anybody."
- Bill Gates

Still, the type of this function lands in $\mathcal{U}_{640001}$ …

Solutions

- Just say $\mathcal{U} : \mathcal{U}$. (Prototype proof assistants everywhere)

- I write $\mathcal{U}$ everywhere and the proof assistant infers the levels. (Coq, LEGO, Idris...)

- Explicitly declare a set of constraints $\mathcal{U}_i < \mathcal{U}_j$. (Matita)

- Prenex (?) level quantification $(\ell : \mathrm{Level}) \to (X : \mathcal{U}_\ell) \to X \to X$ (Agda, Lean)

- Crude but Effective Stratification (McBride)

$$\mathrm{id} : (X : \mathcal{U}_0) \to X \to X$$

$$\mathrm{id}^{\Uparrow} : (X : \mathcal{U}_1) \to X \to X \qquad \text{(hence } \mathrm{id}^{\Uparrow}(\mathcal{U}_0) \text{ works)}$$

Act III

An Order-Theoretic Analysis of Universe Polymorphism

jww Favonia + Reed Mullanix

Generalizing Levels

Instead of $\mathcal{U}_n$ indexed by natural numbers, allow any poset.

$\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 \quad \rightsquigarrow \quad \mathcal{U}_i : \mathcal{U}_j$ for any $i < j$

$A : \mathcal{U}_0 \Rightarrow A : \mathcal{U}_1 \quad \rightsquigarrow \quad A : \mathcal{U}_i \Rightarrow A : \mathcal{U}_j$ for any $i \leq j$

eg $(\mathbb{N}, \leq)$. The usual story.

Generalizing Levels

eg $(\mathbb{Z}, \leq)$. This fixes the id problem without polymorphism!

$$\mathrm{id} : (X : \mathcal{U}_0) \to X \to X$$

Need to plug in a (smaller) universe? Try $\mathcal{U}_{-1}$.

eg $(\mathbb{Q}, \leq)$. Density gives us linear constraints for free:

$$f : (X : \mathcal{U}_{\ell}) \to (Y : \mathcal{U}_{\ell'}) \cdots \quad \text{for } \ell < \ell' \quad \text{(tired)}$$

$$\Rightarrow \quad f : (X : \mathcal{U}_0) \to (Y : \mathcal{U}_1) \cdots \quad \text{(wired)}$$

Generalizing Levels

Thm Type theory with arbitrary (including non-well-founded!) level posets is consistent.

Proof (sketch) The rules of $(L, \leq_L)$-type theory depend only on $\leq_L$ and $<_L$, so given any $f : (L, \leq_L) \to (L', \leq_{L'})$ that preserves the strict order $(x <_L y \Rightarrow f(x) <_{L'} f(y))$ we can translate from L- to L'-type theory.

Any finite L-term mentions only finitely many levels $L' \subseteq L$. But every finite poset L' has a $<$-preserving map $L' \to \mathbb{N}$. $\square$

Generalizing Level~~s~~ Polymorphism

In prenex level quantification, every level context $\Delta = (\ell, \ell' : \mathrm{Level})$ gives rise to a poset of level expressions $H(\Delta)$ in context $(0, 1, \ell, \max(\ell, \ell') \ldots)$.

Laws for <-preserving maps:

- Every level var $\ell \in \Delta$ is a level expr in $H(\Delta)$. $\quad (\Delta \to H(\Delta))$
- An assignment of Δ' vars to Δ vars determines a map from $H(\Delta)$ exprs to $H(\Delta')$ exprs. $\quad ((\Delta \to \Delta') \to H\Delta \to H\Delta')$
- An assignment of $H(\Delta')$ exprs to Δ vars extends to a substitution $H(\Delta) \to H(\Delta')$. $\quad ((\Delta \to H\Delta') \to H\Delta \to H\Delta')$

$\Longrightarrow$ H is a monad on the category of posets and <-preserving maps.

The "McBride monad"

$$M(\Delta) = \Delta \times_0 \mathbb{N}$$

how much to shift up

$$U_\ell^{\Uparrow n} = U_{\ell+n}$$

$$\text{return } \ell = (\ell, 0)$$

$$\text{join } ((\ell, n_1), n_2) = (\ell, n_1 + n_2)$$

The Generalized "McBride monad"

$$M_D(\Delta) = \Delta \ltimes_\star D$$

generalized displacement poset

$$\mathrm{return}\ \ell = (\ell, \star)$$

$$\mathrm{join}\ ((\ell, n_1), n_2) = (\ell, n_1 \bullet n_2)$$

for any "displacement algebra" $(D, \star, \bullet)$

with $x < y \Rightarrow z \bullet x < z \bullet y$

Main Results

<u>Theorem</u> Any $\mathcal{H}$ (generalized level polymorphism) embeds into $\mathcal{M}_D$ (generalized McBride monad) for some D.

In other words, the $\Uparrow^d$ operator can handle any polymorphism scheme!

Drop-in OCaml implementation of generalized $\Uparrow$ at:

github.com/RedPRL/mugen